Math 53, Discussions 116 and 118 **2022-09-28**

Limits

Questions

Compute the following limits, or show they don't exist.

Question 1. $\lim_{(x,y)\to(\pi,\pi/2)} y\sin(x-y)$

Question 2. $\lim_{(x,y)\to(2,-1)} \frac{x^2y+xy^2}{x^2-y^2}$

Question 3. $\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4}$

Question 4. $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

Question 5. $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

Question 6. $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$

Question 7. $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2}$

Question 8. $\lim_{(x,y)\to(0,0)} \frac{x^2y+xy^2}{x^2-y^2}$

Question 9. $\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^4+y^4}$

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to questions

Question 1. Since $y\sin(x-y)$ is continuous everywhere, we can just substitute:

$$\lim_{(x,y)\to(\pi,\pi/2)} y\sin(x-y) = \frac{\pi}{2}\sin(\pi-\pi/2) = \pi/2.$$

Question 2. $\frac{x^2y+xy^2}{x^2-y^2}$ is continuous everywhere $x^2-y^2\neq 0$, so in particular it is continuous at $(2,-1)$. Again we can just substitute, and obtain the answer $-2/3$.

Question 3. If we approach $(0,0)$ along the x-axis, the limit is

$$\lim_{x\to 0}\frac{0}{x^4} = 0.$$

On the other hand, if we approach along the line $y=x$, the limit is

$$\lim_{x\to 0}\frac{x^2\sin^2 x}{2x^4} = \lim_{x\to 0}\frac{\sin^2 x}{2x^2} = \frac{1}{2}.$$

Since we get two disagreeing answers, we conclude that the original 2D limit does not exist.

Question 4. If we switch to polar, we get the equivalent limit

$$\lim_{r\to 0^+,\theta\text{ any}}\frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2} = \lim_{r\to 0^+,\theta\text{ any}} r(\cos^3\theta+\sin^3\theta).$$

Note that

$$-2r \le r(\cos^3\theta+\sin^3\theta) \le 2r$$

for $r\ge 0$. Because $\lim_{r\to 0^+,\theta\text{ any}} -2r = 0 = \lim_{r\to 0^+,\theta\text{ any}} 2r$, we conclude by the Squeeze Theorem that

$$\lim_{r\to 0^+,\theta\text{ any}} r(\cos^3\theta+\sin^3\theta) = 0$$

as well.

Question 5. Same deal as the preceding. If we switch to polar we get

$$\lim_{r\to 0^+,\theta\text{ any}} r^2\ln(r^2).$$

In this case we don't need the Squeeze Theorem, and we can just deal with this as a SVC limit. The answer is 0.

Question 6. If we approach along the line $\langle t,0,0\rangle$, we get the limit of 0. If we approach along the line $\langle t,t,0\rangle$, we get 1/2. So the limit doesn't exist.

Question 7. You can apply the Squeeze Theorem to show that this limit is zero, by observing that

$$0 \le \frac{x^2}{x^2+y^2+z^2} \le 1$$

and so

$$0 \le \frac{x^2y^2z^2}{x^2+y^2+z^2} \le y^2z^2$$

where both bounds tend to 0 as we approach the origin.

The last two questions are not great because I came up with them spontaneously (the previous ones were from the textbook). Don't worry about them.

Question 8. If you try along any line through the origin other than $y=\pm x$ you get the answer of zero. But the suspicion is that the expression should blow up along these two lines. So let's try along the parabola $y=x-x^2$ (which is tangent to $y=x$ at $(0,0)$). We get

$$\lim_{x\to 0}\frac{x^2(x-x^2)+x(x-x^2)}{x^2-(x-x^2)^2} = \lim_{x\to 0}\frac{x^2-1}{x^2-2x}$$

which DNE, confirming our suspicions. So the original limit does not exist.

Question 9. After applying the change of variables $u = x^2, v = y^2$, this is the same limit as

$$\lim_{(u,v)\to(0,0)} \frac{u^{3/2}v^{3/2}}{u^2 + v^2}.$$

You can show that this limit is zero by switching to polar via $u = r\cos\theta$ and $v = r\sin\theta$ and using the Squeeze Theorem, just as in Question 4.